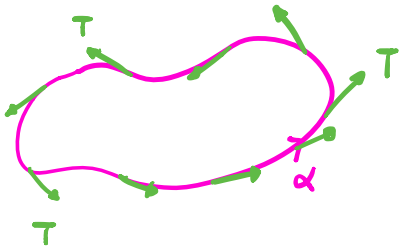


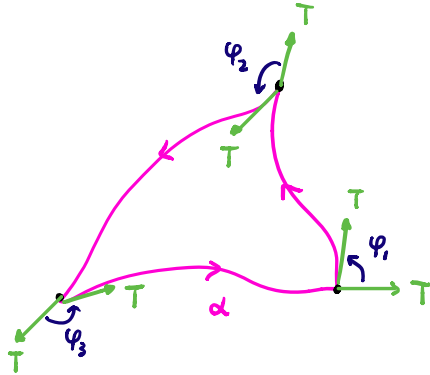
§ The Gauss-Bonnet Theorems

Recall: Theorem of Turning Tangents



$$\int_{\alpha} k(s) ds = 2\pi$$

It also works for "piecewise smooth" curves:



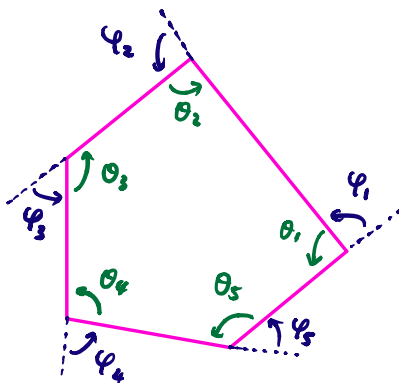
$$\int_{\alpha} k(s) ds + \sum_i \varphi_i = 2\pi \quad (*)$$

Special case: When all the smooth pieces are **geodesics**

(*) \Rightarrow exterior angle sum of polygons = 2π



interior angle sum of n -gon = $(n-2)\pi$



i.e. $\sum_{i=1}^n \varphi_i = 2\pi$, $\sum_{i=1}^n \theta_i = (n-2)\pi$.

Question: What about surfaces? Is the "total curvature" of a surface a constant?

Gauss Bonnet Theorem I:

Let S be a compact orientable surface without boundary.

Then

$$\int_S K da = 2\pi \chi(S)$$

where $\chi(S)$ is the Euler characteristics of S

given by the formula $\chi(S) = 2 - 2g$.
↑ genus of S

Examples:



...



genus:

0

1

2

...

g

χ :

2

0

-2

...

$2-2g$

> 0

$= 0$

< 0

Note: The Gauss-Bonnet Theorem is a beautiful result since it relates "geometry" on L.H.S. to "topology" on R.H.S.

There is a version for surfaces with boundary.

Gauss-Bonnet Theorem II:

Let S be a compact orientable surface with boundary.

Then,

$$\int_S K da + \int_{\partial S} k_g ds = 2\pi \chi(S)$$

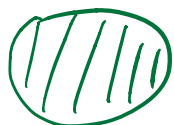
$\partial S \leftarrow$ positively oriented

where $\chi(S) = 2 - 2g - \gamma$

$g =$ genus

$\gamma =$ # of boundaries

Example:



g :

0

0

0

γ :

1

2

3

χ :

1

0

-1



> 0



$= 0$



< 0

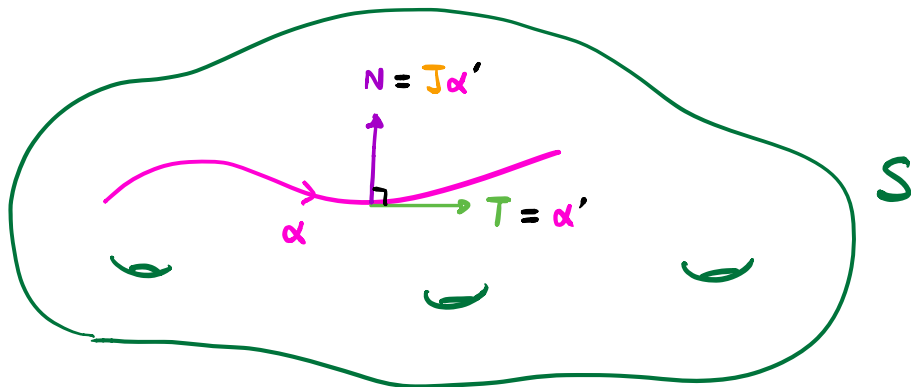
Definition: Let S be an oriented surface, then

$\exists J : T_p S \rightarrow T_p S$ rotation by 90° on each tangent plane.

If $\alpha : I \rightarrow S$ is a curve p.b.a.l.,

we define the geodesic curvature of α

$$k_g := \langle \nabla_{\alpha'} \alpha', J \alpha' \rangle$$



Remark: (1) If $S = \text{plane}$ (with standard orientation)

then k_g agrees with the usual curvature k for plane curves.

(2) Switch the orientation of α changes the sign of k_g

(3) α geodesic $\Leftrightarrow k_g \equiv 0$

(i.e. geodesic curvature measures how much is the curve deviated from a geodesic.)

There is also a version of Gauss-Bonnet Theorem which allows the boundary to be only "piecewise smooth":

$$\int_S K da + \int_{\partial S} k_g ds + \underbrace{\sum_i \varphi_i}_{\text{sum of exterior angles}} = 2\pi \chi(S)$$

Gauss-Bonnet Theorem II

sum of exterior angles.

§ Applications of Gauss-Bonnet Theorems

(I) Any compact orientable surface S without boundary and $K > 0$ everywhere is homeomorphic to a sphere.

Proof: By Gauss-Bonnet I,

$$\int_S K da = 2\pi \chi(S)$$

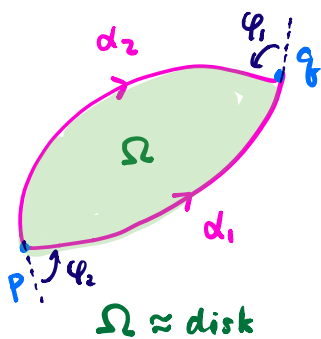
Since $K > 0$ everywhere, the L.H.S. > 0 , hence

$$\chi(S) > 0 \Rightarrow S \approx \text{sphere.}$$

_____ ◻

(II) If S is a surface with $K \leq 0$ everywhere, then any two geodesics α_1, α_2 starting at the same point $p \in S$ cannot meet again at some $q \in S$ s.t. α_1, α_2 together bounds a "disk".

Proof: By contradiction. Suppose not.



Apply Gauss-Bonnet III to Ω

$$\int_{\Omega} K da + \int_{\partial\Omega} k_g ds + \varphi_1 + \varphi_2 = 2\pi \chi(\Omega)$$

Since $\partial\Omega = \alpha_1 \cup \alpha_2$ are geodesics

$$\Rightarrow k_g \equiv 0.$$

Since $\Omega \approx \text{disk}$

$$\Rightarrow \chi(\Omega) = 1$$

Therefore,

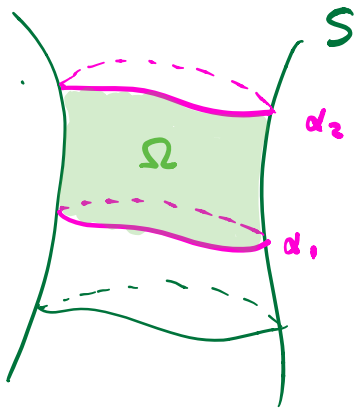
$$\underbrace{\int_{\Omega} K da}_{\hat{0}} + \underbrace{\varphi_1}_{\hat{\pi}} + \underbrace{\varphi_2}_{\hat{\pi}} = 2\pi$$

Contradiction!

_____ \square

(II) If S is homeomorphic to a cylinder and $K < 0$ everywhere on S then \exists at most 1 simple closed geodesic on S

Sketch of Proof: Suppose NOT. We have 2 such geodesics α_1, α_2 which together must bound a cylinder in S



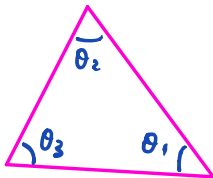
Apply Gauss-Bonnet II to Ω

$$\underbrace{\int_{\Omega} K da}_{< 0} + \int_{\partial\Omega} k_g ds = 2\pi \chi(\Omega)$$

Contradiction!

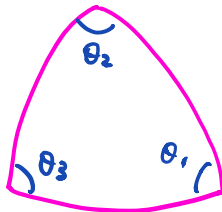
(IV) Interior angle sum of "geodesic triangles"

$K \equiv 0$



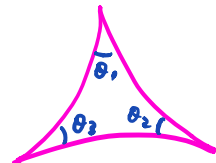
$\sum \theta_i = \pi$

$K > 0$



$\sum \theta_i > \pi$

$K < 0$



$\sum \theta_i < \pi$